UNIT-II

Magnetostatics :

1. Magnetic Field Intensity
2. Magnetic Flux & Magnetic Flux Density
3. Biot-Savart's law
4. Ampere's Circuit Law
5. Applications of Ampere's circuit law
6. Maxwell's two equations for magnetostatic fields
7. Magnetic Scalar & Vector Potentials
8. Force due to magnetic field
    a) Force on a charged Particle
    b) Force on a current element
    c) Force between two current elements.
9. Amperes Force law
10. Concept of Inductance
11. Magnetic Energy & Energy Density.

UNIT-II YVNR MF-1

MAGNETIC FIELD

In the year 1820, the Scientist Hans Christian Orsted established the relationship between electric and magnetic fields. He observed that, an electrostatic field is produced by Static (or) stationary charges (electrons). When the charges are in motion, they produce the magnetic field around them. If the charges are moving with a constant velocity a static magnetic field (or) magnetostatic field is produced.

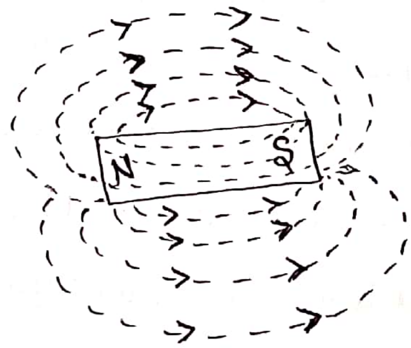
→ A magnetostatic field is produced by a constant current flow (or) direct current. In this chapter, we consider magnetic fields in free space due to direct current.

→ There are two types of magnets, permanent magnet and electromagnet

* Permanent magnets are made from iron, steel, nickel (or) cobalt by electronic orientation.
* The current elements which provide magnetic fields are called electromagnets

Permanent magnet:

The two opposit ends of a magnet are called poles. One is north (N) pole and other is South (S) pole. as shown in fig.



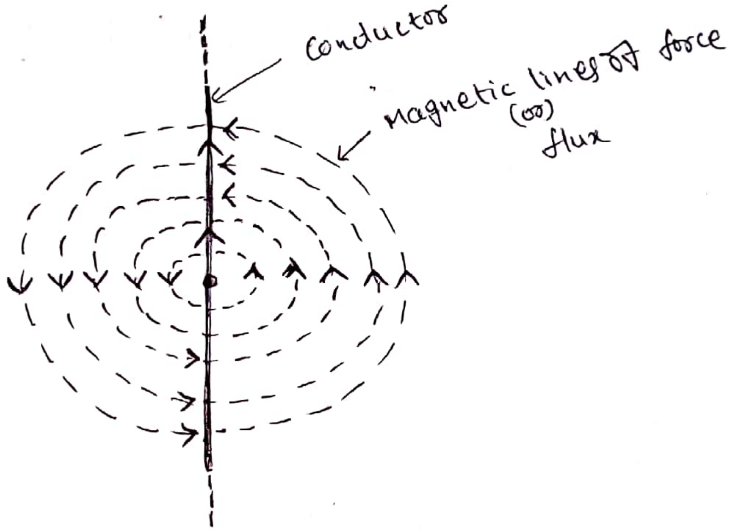
→ The magnetic field is represented by the imaginary lines around the magnet which are called magnetic lines (or) magnetic flux.

→ The flux lines always flow from North pole to South pole in the space and from S pole to N pole inside the magnet. So a single pole magnet never exists.

→ Many devices in our every day life such as motors, transformers, telephones, television, microphones, compasses, CRO displays, Storage devices etc., are based on electromagnetic principles.

Magnetic Field due to Current carrying Conductor :

When a conductor carries a current, it produces a magnetic field arount it along the conductor.



→ The magnetic flux lines are in the form of concentric circles on the plane perpendicular to the conductor as shown in fig.

→ If the current is constant i.e., direct current, a steady magnetic field exists around the conductor.

* The region around a magnet within which the influence of the magnet can be experienced is called magnetic field.

Magnetic Field Intensity (H):

Magnetic field intensity at any point is defined as the force experienced by a unit north pole at that point when it is placed in a uniform magnetic field.

→ It is denoted with H and the units are Newtons (N)/weber (wb) (or) Amper turn /meter. (AT/m).

→ It is a vector quantity and is analogous to electric field Intensity.

Magnetic flux (ψ) :

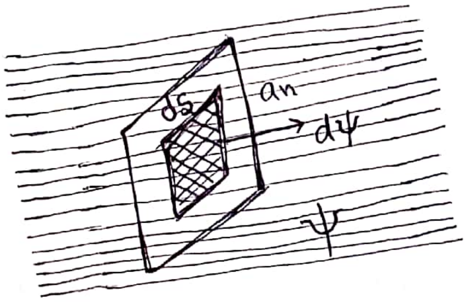
In a magnetic field, the magnetic lines of force always exist in the form of concentric circles.

* The total number of lines of force in a magnetic field is called a magnetic flux.

→ It is denoted with ψ. and the units are webers (Wb).

Magnetic flux Density (B) :

In the magnetic field, the magnetic flux crossing through the unit area normal to the direction of flux is called magnetic flux density.



→ It is ~~also called~~ denoted with B and the units are webers per square meter (Wb/m^2) (or) Tesla (T).

→ If the differential flux $d\psi$ crosses the differential area dS normal to the plane of the loop as shown in fig. then the magnetic flux density is given by

$$\boxed{B = \frac{d\psi}{dS} a_n} \quad Wb/m^2 \text{ (or) Tesla}$$

∴ The magnetic flux is given by

$$d\psi = B \cdot dS \Rightarrow \boxed{\psi = \int_S B \cdot dS}$$

* The net magnetic flux passing through a unit surface area is called magnetic flux density B.

Relationship between B & H :

→ The magnetic flux density (B) is related with magnetic field intensity (H) through the material property called permeability (μ) i.e.,

$$B = \mu H$$

$$B = \mu_0 \mu_r H$$

Here μ_r → relative permeability & μ_0 → Absolute permeability

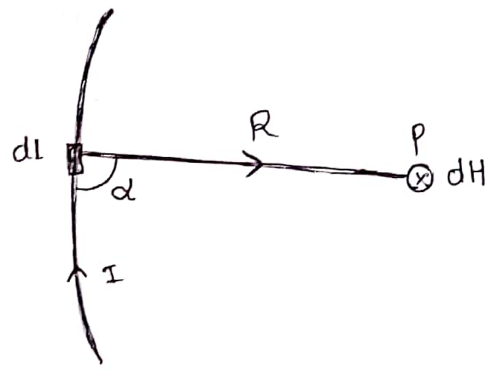
→ for free space $\mu_r = 1$ then $\mu = \mu_0 = 4\pi \times 10^{-7}$ H/m

→ for magnetic materials μ_r is greater than 1 ($\mu_r > 1$)

* for non magnetic material μ_r is equal to 1 ($\mu_r = 1$).

Biot-Savart's law :

Consider a conductor carrying a direct current I and steady magnetic field produced around it.



→ Biot-Savarts law allows us to obtain the differential magnetic field Intensity dH, produced at Point P, due to a differential current element Idl. The current carrying conductor is shown in fig.

* The Biot-Savart's law States that the differential magnetic field intensity dH produced at a Point due to a current element is

1. Proportional to the product of the current I and differential length dl. (current element Idl) &
2. The Sin of the angle α between the element and the line Joining Point P to the element
3. And inversely proportional to the square of the distance R between Point P and the element.

→ Expressed Mathematically $dH \propto \frac{I dl \sin\alpha}{R^2}$

$$dH = \frac{K I dl \sin\alpha}{R^2}$$

where K is proportionality constant. Its value $K = \frac{1}{4\pi}$, then

$$dH = \frac{I dl \sin\alpha}{4\pi R^2}$$

$$dH = \frac{I dl \times a_R}{4\pi R^2}$$

$$\boxed{dH = \frac{I dl \times R}{4\pi |R|^3}}$$

from Cross product

$dl \times a_R = dl\, |a_R| \sin\theta$
$= dl \sin\theta$

$a_R = \frac{R}{|R|}$

∴ The total magnetic field intensity is

$$\boxed{H = \int_L \frac{I dl \times R}{4\pi |R|^3}}$$

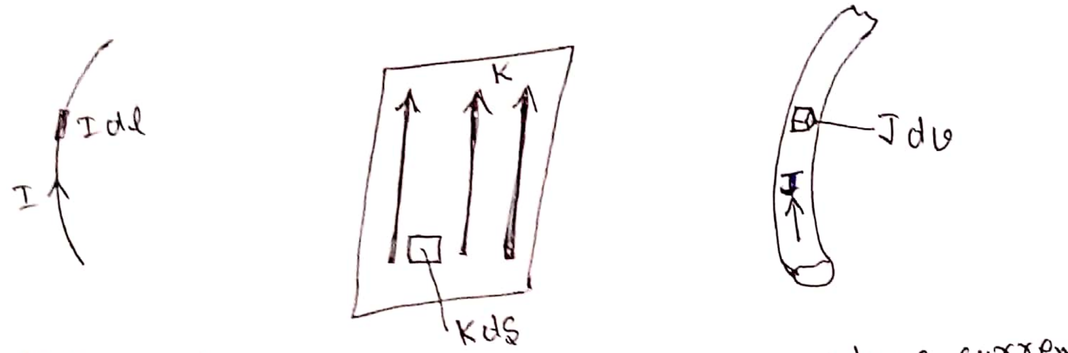
* The direction of magnetic field can be determined by right handed thumb rule (or) right handed screw rule.

Different Current Distributions:

→ The line current, Surface current, and volume current as shown in fig.

→ If we define K as the Surface current density in A/m and J as the volume current density in A/m^2.

→ The Source elements are related as $\boxed{I\,dl \equiv K\,dS \equiv J\,dv}$

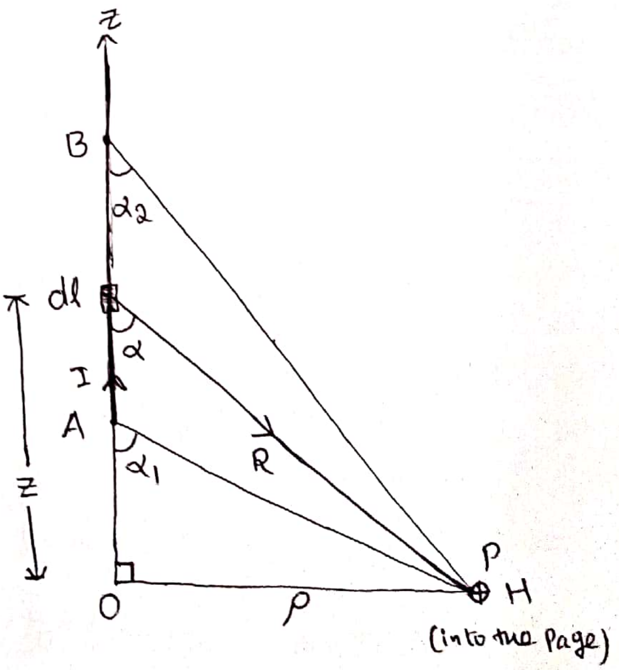


line current — Surface current — Volume current

$$H = \int_L \frac{I\,dl \times a_R}{4\pi R^2} \qquad H = \int_S \frac{K\,dS \times a_R}{4\pi R^2} \qquad H = \int_V \frac{J\,dv \times a_R}{4\pi R^2}$$

Magnetic Field due to a finite Current element:

Assume that the conductor is along the z-axis with its upper and lower ends respectively subtending angles α_2 and α_1 at P, the point at which H is to be determined.



→ Consider the differential magnetic field dH at P due to an element dl at (0,0,z).

$$dH = \frac{I\,dl \times R}{4\pi\,|R|^3}$$

But $dl = dz\,a_z$ and

$R = \rho a_\rho - z a_z$

$|R| = \sqrt{\rho^2 + z^2}$

$\therefore \; dl \times R = dz\, a_z \times (\rho a_\rho - z a_z)$

$= \rho\, dz (a_z \times a_\rho) - z\, dz (a_z \times a_z)$

$= \rho\, dz\, a_\phi - z\, dz\, (0)$

$= \rho\, dz\, a_\phi$

$a_\rho \times a_\phi = a_z$

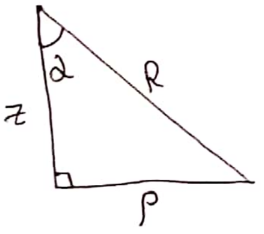
$a_\phi \times a_z = a_\rho$

$a_z \times a_\rho = a_\phi$

$a_z \times a_z = 0$

$1 + \cot^2\theta = \operatorname{cosec}^2\theta$

$$\therefore H = \int \frac{I\, dl \times R}{4\pi |R|^3} = \int \frac{I \rho\, dz}{4\pi [\rho^2 + z^2]^{3/2}}\, a_\phi$$



$\tan\alpha = \frac{\rho}{z} \Rightarrow \cot\alpha = \frac{z}{\rho} \Rightarrow z = \rho\cot\alpha$

$\Rightarrow dz = -\rho \operatorname{cosec}^2\alpha\, d\alpha$

$$\therefore H = \int \frac{I\rho\, dz\; a_\phi}{4\pi(\rho^2 + z^2)^{3/2}} = \int_{\alpha_1}^{\alpha_2} \frac{I\rho(-\rho\operatorname{cosec}^2\alpha\, d\alpha)}{4\pi(\rho^2 + \rho^2\cot^2\alpha)^{3/2}}\, a_\phi$$

$$H = \frac{-I}{4\pi}\int_{\alpha_1}^{\alpha_2} \frac{\rho^2 \operatorname{cosec}^2\alpha\, d\alpha}{[\rho^2\operatorname{cosec}^2\alpha]^{3/2}}\, a_\phi = \frac{-I}{4\pi}\int_{\alpha_1}^{\alpha_2} \frac{\rho^2\operatorname{cosec}^2\alpha\, d\alpha}{\rho^3\operatorname{cosec}^3\alpha}\, a_\phi$$

$$H = \frac{-I}{4\pi\rho}\int_{\alpha_1}^{\alpha_2} \frac{1}{\operatorname{cosec}\alpha}\, d\alpha\, a_\phi = \frac{-I}{\rho 4\pi}\int_{\alpha_1}^{\alpha_2} \sin\alpha\, d\alpha\, a_\phi$$

$$H = \frac{-I}{4\pi\rho}\left[-\cos\alpha\right]_{\alpha_1}^{\alpha_2} a_\phi = \frac{I}{4\pi\rho}\left[\cos\alpha_2 - \cos\alpha_1\right] a_\phi$$

$$\boxed{\therefore H = \frac{I}{4\pi\rho}\left[\cos\alpha_2 - \cos\alpha_1\right] a_\phi} \longrightarrow (1)$$

→ This expression is generally applicable for any straight filamentary conductor of finite length.

→ From the above equation H is always along the unit vector a_ϕ (i.e., along concentric circular paths) irrespective of the length of the wire (or) the point of interest P.

MF-4

Case 1 :

→ When the conductor is semifinite (with respect to P) so that point A is now at O (0,0,0) while B is at (0,0,∞); $\alpha_1 = 90°$, $\alpha_2 = 0°$, then eq (1) becomes.

$$H = \frac{I}{4\pi\rho}[\cos 0 - \cos 90]a_\phi = \frac{I}{4\pi\rho}[1-0]a_\phi = \frac{I}{4\pi\rho}a_\phi$$

$$\boxed{\therefore H = \frac{I}{4\pi\rho}a_\phi}$$

Case 2 :

→ When the conductor is infinite in length. For this case, point A is at (0,0,−∞) while B is at (0,0,∞); $\alpha_1 = 180°$, $\alpha_2 = 0°$ then eq (1) becomes

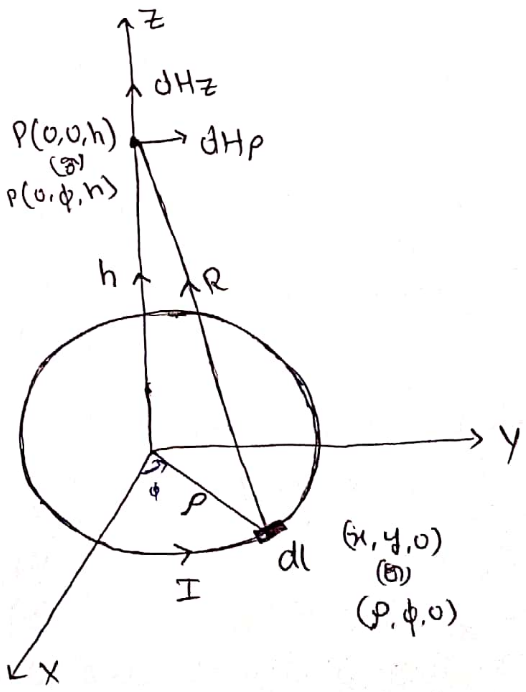
$$H = \frac{I}{4\pi\rho}[\cos 0 - \cos 180]a_\phi = \frac{I}{4\pi\rho}[1-(-1)]a_\phi = \frac{I}{4\pi\rho}2a_\phi = \frac{I}{2\pi\rho}a_\phi$$

$$\boxed{\therefore H = \frac{I}{2\pi\rho}a_\phi}$$

Problem A circular loop located on $x^2+y^2=9$, $z=0$ carries a direct current of 10A along a_ϕ. Determine H at (0,0,4) and (0,0,−4).

Sol:

• Consider the circular loop shown in fig.



→ The magnetic field intensity dH at point P(0,0,h) contributed by current element Idl is given by Biot-Savart's law

$$dH = \frac{I\,dl \times R}{4\pi |R|^3}$$

Consider a cylindrical coordinates

where $dl = \rho\, d\phi\, a_\phi$.

$$R = (0,\phi,h) - (\rho,\phi,0)$$

$$R = -\rho a_\rho + h a_z$$

$$|R| = \sqrt{\rho^2+h^2}$$

then $dl \times R = \rho\, d\phi\, a_\phi \times (-\rho a_\rho + h a_z)$

$$= -\rho^2 d\phi (a_\phi \times a_\rho) + h\rho\, d\phi (a_\phi \times a_z)$$

$$d\vec{l} \times \vec{R} = -\rho^2 d\phi(-a_z) + \rho h\, d\phi\, a_\rho$$

$$d\vec{l} \times \vec{R} = \rho^2 d\phi\, a_z + \rho h\, d\phi\, a_\rho$$

Hence $$dH = \frac{I(\rho^2 d\phi\, a_z + \rho h\, d\phi\, a_\rho)}{4\pi(\rho^2+h^2)^{3/2}}$$

$$H = \int_L \frac{I}{4\pi} \frac{(\rho^2 d\phi\, a_z + \rho h\, d\phi\, a_\rho)}{(\rho^2+h^2)^{3/2}}$$

→ Since the loop is symmetrical along the radial direction, the a_ρ component is zero.

$$H = \frac{I}{4\pi}\int_L \frac{\rho^2 d\phi\, a_z}{(\rho^2+h^2)^{3/2}}$$

→ For circular loop, the limits are 0 to 2π

$$\therefore H = \frac{I}{4\pi}\int_0^{2\pi} \frac{\rho^2 d\phi\, a_z}{(\rho^2+h^2)^{3/2}} = \frac{I\rho^2 a_z}{4\pi(\rho^2+h^2)^{3/2}}\int_0^{2\pi} d\phi$$

$$H = \frac{I\rho^2 a_z}{\cancel{4\pi}_{2}(\rho^2+h^2)^{3/2}}\,\cancel{2\pi} = \boxed{\frac{I\rho^2 a_z}{2[\rho^2+h^2]^{3/2}} = H}$$

(a) Substituting $I = 10\,A$, $\rho = 3$ and $h = 4$ gives

$$H(0,0,4) = \frac{10(3)^2 a_z}{2(9+16)^{3/2}} = \frac{90}{2(5^3)} a_z = 0.36 a_z \text{ A/m}$$

(b) If h is replaced by $-h$, the z-component of dH remains the same while the ρ-component still adds up to zero due to the axial symmetry of the loop. Hence $H(0,0,-4) = H(0,0,4) = 0.36 a_z$ A/m.

* "The position vector at current element (dl) is $(\rho, \phi, 0)$ and the position vector at point P is $(0, \phi, z)$, then distance vector is

$$R = (0,\phi,z) - (\rho,\phi,0) = -\rho a_\rho + 0 a_\phi + z a_z = -\rho a_\rho + z a_z$$"

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MF-5

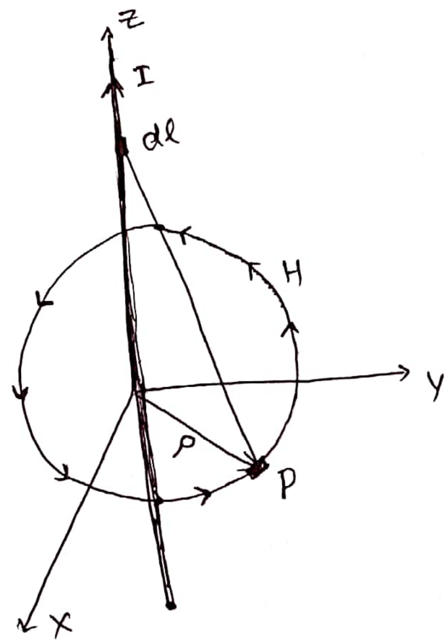
Ampere's Circuit law (or) Ampere's work law :

Ampere's Circuit law States that, the line integral of magnetic field intensity H around a closed Path is equal to the net current I_{encl} enclosed by that Path.

It is mathematically expressed as

$$\boxed{\oint_L H \cdot dl = I_{encl}} \text{ Amps.}$$

Proof :



Consider a Long conductor carrying current I along z-axis as shown in fig.

→ Consider a closed circular Path of radius ρ around the conductor with current element $I\,dl$.

→ Using Biot-Savart's Law, the H at Point P due to infinite length current is $H = \frac{I}{2\pi\rho} a_\phi$ and at Point P dl is in a_ϕ direction

i.e., $dl = \rho\, d\phi\, a_\phi$

then $H \cdot dl = \frac{I}{2\pi\rho} a_\phi \cdot \rho\, d\phi\, a_\phi = \frac{I}{2\pi\rho} \rho\, d\phi\,(1) = \frac{I}{2\pi} d\phi$

→ The line Integral of closed Path is

$$\oint_L H \cdot dl = \int_0^{2\pi} \frac{I}{2\pi} d\phi = \frac{I}{2\pi} 2\pi = I.$$

→ If $I = I_{encl}$, the current enclosed by the Path, then

$$\boxed{\therefore \oint_L H \cdot dl = I_{encl}}$$

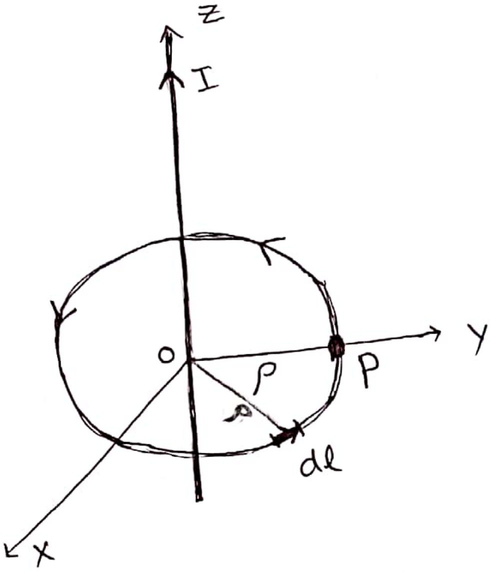
→ Ampere's law is similar to Gauss's law and it is easily applied to determine H when the current distribution is symmetrical.

→ Ampere's law is used only for symmetric current distributions. It is the limitation of Ampere's law.

Applications of Ampere's law :

Now apply Ampere's circuit law to determine H for symmetric current distributions. We will consider an infinite line current, infinite sheet current and an infinitely long coaxial transmission line.

1. Infinite Line Current :



Consider an infinitely long straight conductor placed along z-axis, carrying a direct current I as shown in fig.

→ Consider a point P on the closed path at which H is to be determined

→ The radius of the path is ρ and hence P is at a perpendicular distance ρ from the conductor.

→ The magnitude of H depends on ρ and the direction is always tangential to the closed path i.e., a_ϕ. So H has only component in a_ϕ direction say H_ϕ.

→ Consider elementary length dl at point P and in cylindrical co-ordinates it is $\rho\, d\phi$ in a_ϕ direction.

$$\therefore H = H_\phi a_\phi \quad \text{and} \quad dl = \rho\, d\phi\, a_\phi$$

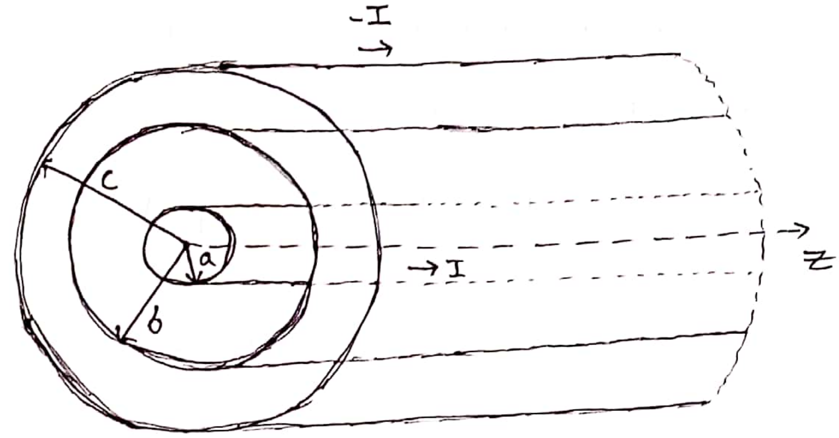
$$\therefore H \cdot dl = H_\phi a_\phi \cdot \rho\, d\phi\, a_\phi = H_\phi \rho\, d\phi$$

→ According to Amperes circuit law

$$\oint H \cdot dl = I \Rightarrow \int_0^{2\pi} H_\phi \rho\, d\phi = I \Rightarrow H_\phi \rho . 2\pi \Rightarrow H_\phi = \frac{I}{2\pi\rho}$$

$$\therefore H = H_\phi a_\phi \Rightarrow \boxed{H = \frac{I}{2\pi\rho} a_\phi}$$

2. Co-axial cable :



Consider a co-axial cable as shown in fig.

→ Its inner conductor is solid with radius 'a', carrying direct current I.

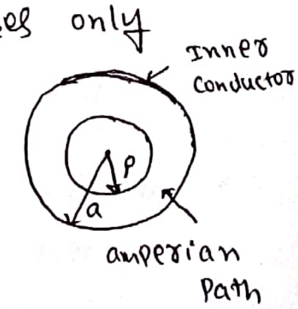
→ The outer conductor is in the form of concentric cylinder whose inner radius is 'b' and outer radius is 'c'.

→ This cable is placed along z-axis. The current I is uniformly distributed in the inner conductor. While -I is uniformly distributed in the outer conductor.

→ The space between inner and outer conductor is filled with dielectric say air. The calculation of H is divided corresponding to various regions of the cable.

Region 1 : With in the inner conductor, $\rho < a$.

→ Consider a closed path having radius $\rho < a$. Hence it encloses only part of the conductor as shown in fig.



→ The total current I is flowing through the cross section area $\pi a^2 = I \longrightarrow (1)$

→ While the closed path encloses the cross section area $\pi \rho^2 = I' \longrightarrow (2)$.

To get I' divid (2) with (1)

$$\therefore \frac{I'}{I} = \frac{\pi \rho^2}{\pi a^2} \Rightarrow I' = \frac{\pi \rho^2}{\pi a^2} I \Rightarrow I' = I_{encl} = \frac{\rho^2}{a^2} I.$$

→ The H is only in a_ϕ direction and depends only on ρ.

$\therefore H = H\phi a\phi$ and $dl = \rho d\phi a\phi$

$\therefore H \cdot dl = H\phi a\phi \cdot \rho d\phi a\phi = H\phi \rho d\phi$

→ According to ampere's circuit law

$$\oint H \cdot dl = I' \Rightarrow \int_0^{2\pi} H\phi \rho d\phi = \frac{\rho^2}{a^2} I$$

$$\therefore \int_0^{2\pi} d\phi = [\phi]_0^{2\pi} = 2\pi$$

$$\Rightarrow H\phi \rho 2\pi = \frac{\rho^2}{a^2} I$$

$$H\phi = \frac{I\rho}{2\pi a^2}$$

$$\therefore H = H\phi a\phi \Rightarrow \boxed{H = \frac{I\rho}{2\pi a^2} a\phi} \quad A/m$$

Region 2 : Within $a < \rho < b$.

→ Consider a closed path (or) circular path which encloses the inner conductor carrying direct current I.

→ This is the case of infinite long conductor along z-axis.

Hence H in this region is

$$\boxed{H = \frac{I}{2\pi\rho} a\phi} \quad A/m$$

Region 3 : Within outer conductor $b < \rho < c$.

→ The current enclosed by the closed path is only the part of the current $-I$, in the outer conductor.

→ The total current $-I$ is flowing through the cross section area given by $-I = \pi(c^2 - b^2) \longrightarrow (3)$

→ While the closed path encloses the cross section area given by $I' = \pi(\rho^2 - b^2) \longrightarrow (4)$

To get I' divid eq (4) with eq (3)

$$\frac{I'}{-I} = \frac{\pi(\rho^2 - b^2)}{\pi(c^2 - b^2)} \Rightarrow I' = \frac{(\rho^2 - b^2)}{(c^2 - b^2)}(-I) \Rightarrow I' = \frac{-(\rho^2 - b^2)}{(c^2 - b^2)} I.$$

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MF-7

→ The current in the inner conductor is $I'' = I$

$\therefore$ The current enclosed by the path is $I_{encl} = I' + I''$

$$I_{encl} = \frac{-(\rho^2 - b^2)}{(c^2 - b^2)} I + I = I\left[1 - \frac{\rho^2 + b^2}{c^2 - b^2}\right] = I\left[\frac{c^2 - b^2 - \rho^2 + b^2}{c^2 - b^2}\right]$$

$$I_{encl} = I\left[\frac{c^2 - \rho^2}{c^2 - b^2}\right]$$

→ The H is only in a_ϕ direction and depends only on ρ (radius).

$$\therefore H \cdot dl = H_\phi a_\phi \cdot \rho d\phi a_\phi = H_\phi \rho d\phi.$$

→ According to Ampere's circuit law

$$\oint H \cdot dl = I_{encl} \Rightarrow \int_0^{2\pi} H_\phi \rho d\phi = I\left[\frac{c^2 - \rho^2}{c^2 - b^2}\right]$$

$$\Rightarrow H_\phi \rho 2\pi = I\left[\frac{c^2 - \rho^2}{c^2 - b^2}\right]$$

$$\Rightarrow H_\phi = \frac{I}{2\pi\rho}\left[\frac{c^2 - \rho^2}{c^2 - b^2}\right]$$

$$\therefore H = H_\phi a_\phi \Rightarrow \boxed{H = \frac{I}{2\pi\rho}\left[\frac{c^2 - \rho^2}{c^2 - b^2}\right] a_\phi} \quad A/m.$$

Region 4 : Out side the cable $\rho > c$.

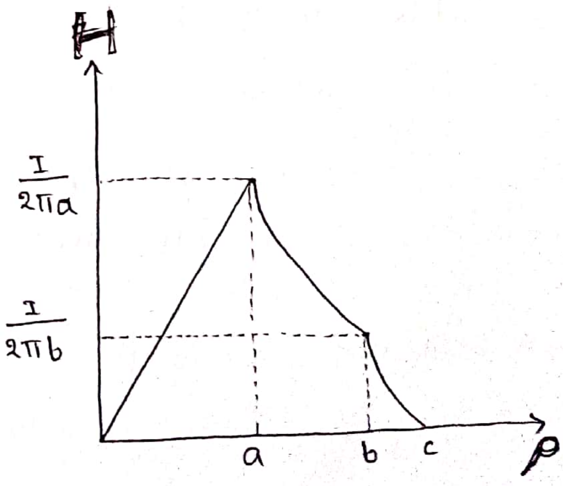
→ Consider a closed path with $\rho > c$ such that it encloses both the conductors i.e., both currents $+I$ and $-I$.

→ Thus current enclosed is $I_{encl} = +I - I = 0$

→ According to Ampere's law $\oint H \cdot dl = I_{encl}$

$$\therefore \oint H \cdot dl = 0$$

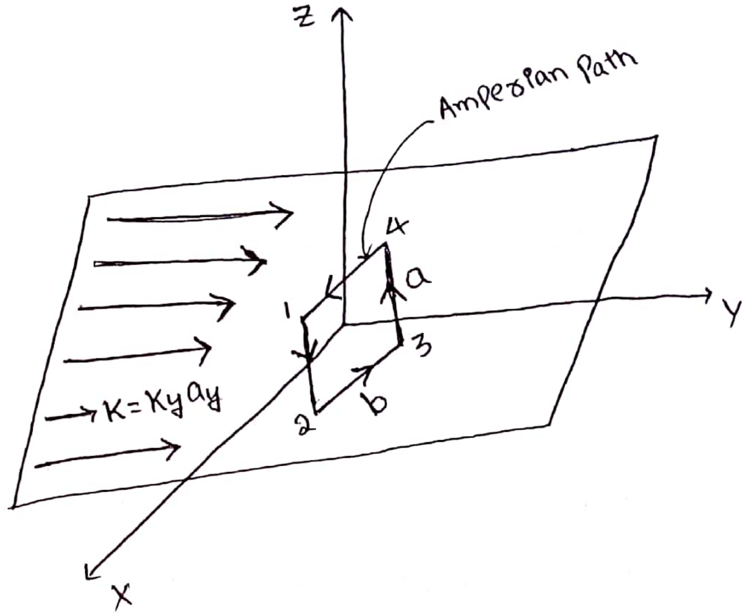
$$\boxed{\therefore H = 0} \quad A/m$$



→ The magnetic field does not exist out side the cable

→ Variation of magnetic field H with radius ρ of the coaxial cable as shown in graph.

3. Infinite Sheet of Current :



→ Consider an infinite sheet of current on x-y plane at $z = 0$.

→ The current is flowing in the positive y-direction hence the surface current density is $K = K_y a_y$ A/m.

→ Consider a closed path (Amperian path) 1-2-3-4 as shown in fig. with width 'b' and height 'a'. It is perpendicular to the direction of current hence in the x-z plane.

→ The current flowing across the width 'b' is given by

$$I_{encl} = K_y b$$

→ From the Ampere's circuit law

$$\oint H \cdot dl = I_{encl} = K_y b \longrightarrow (1)$$

→ To evaluate the integration, the symmetrical conditions should be considered. Since the sheet is extended to infinity on x-y plane, due to symmetry, the field along z-direction is zero. As current flow in y-direction, the component H_y is also zero.

∴ H has only component in x-direction.

i.e.,

$$H = H_x a_x \quad \text{for } z > 0$$
$$H = -H_x a_x \quad \text{for } z < 0$$

→ Evaluating line integral along the closed path 1-2-3-4-1 using Ampere's circuit law.

$$\oint H \cdot dl = I_{encl}.$$

MF-8

$$\oint H\cdot dl = \left(\int_1^2 + \int_2^3 + \int_3^4 + \int_4^1\right) H\cdot dl$$

$$= 0(-a) + (-H_x)(-b) + 0(a) + H_x(b)$$

$$= 0 + H_x b + 0 + H_x b$$

$$\therefore \oint H\cdot dl = 2H_x b \longrightarrow (2)$$

→ Equating eq (1) & eq (2)

$$2H_x b = K_y b$$

$$H_x = \frac{1}{2} K_y$$

Hence $H = H_x a_x = \frac{1}{2} K_y a_x$ for $z > 0$

$H = -H_x a_x = -\frac{1}{2} K_y a_x$ for $z < 0$

→ In general, for an infinite sheet of current density K A/m we can write

$$\boxed{H = \frac{1}{2} K \times a_n}$$

→ Where a_n is the unit vector normal to the direction of current flow on the sheet

Maxwell's Two equations for Magneto Static Fields :

3rd Equation :

Curl of the magnetic field intensity H is equal to the current density J. i.e., $\boxed{\nabla \times H = J}$

Proof : From the definition of Ampere's circuit law

$$\oint_L H \cdot dl = I$$

Applying Stokes theorem to the left-hand side of above eq.

$$\int_S (\nabla \times H) \cdot dS = I \longrightarrow (1)$$

→ But we know the current density $J = \frac{\Delta I}{\Delta S}$

$$\therefore J = \frac{dI}{dS} \Rightarrow dI = J \cdot dS$$

$$\Rightarrow I = \int_S J \cdot dS \longrightarrow (2)$$

Equating eq (1) & (2).

$$\int_S (\nabla \times H) \cdot dS = \int_S J \cdot dS$$

Hence $\boxed{\nabla \times H = J}$ This is the Maxwell's third equation.

4th Equation :

Divergence of the magnetic flux density B is equal to zero.

i.e., $\boxed{\nabla \cdot B = 0}$

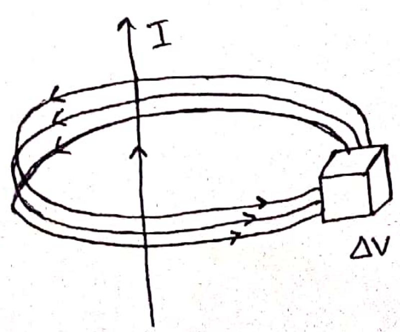


fig: H due to straight conductor

Proof :

The divergence of any flux density vector field is the net out flow of flux per unit volume.

→ We know that the flux due to a current carrying conductor is in the form of concentric circles.

→ Consider an infinitesimal volume ΔV as shown in fig.
The number of flux lines entering and leaving the volume ΔV are equal.
Therefore the net outflow of flux per unit volume is zero.

→ No magnetic flux can reside in a closed surface. Hence the integral $B \cdot ds$ evaluated over a closed surface is always zero.

$$\therefore \quad \oint_S B \cdot ds = 0$$

→ This is called law of conservation of magnetic flux.
Applying divergence theorem to the above eq.

$$\oint_S B \cdot ds = \int_V (\nabla \cdot B)\, dv = 0$$

$$\Rightarrow \boxed{\nabla \cdot B = 0}$$

(or)

⇒ The total flux through any closed surface within the magnetic field B is zero.

$$\oint_S B \cdot ds = \int_V (\nabla \cdot B)\, dv = 0 \Rightarrow \boxed{\nabla \cdot B = 0}$$

Maxwell's Equations for Static EM Fields:

S.No	Differential (Point) form	Integral form	Remarks
1.	$\nabla \cdot D = \rho_v$	$\oint_S D \cdot ds = \int_V \rho_v \, dv = Q$	Gauss's law.
2.	$\nabla \times E = 0$	$\oint_L E \cdot dl = 0$	Conservation of electric field.
3.	$\nabla \times H = J$	$\oint_L H \cdot dl = \int_S J \cdot ds = I$	Ampere's circuit law.
4.	$\nabla \cdot B = 0$	$\oint_S B \cdot ds = 0$	Conservation of magnetic flux.

Magnetic Scalar and Vector Potentials :

1. Magnetic Scalar Potential : (V_m)

→ We know the potential in Electrostatic field is $V = -\int E \cdot dl$

and relation b/w E & V is $E = -\nabla V$.

→ Similarly, we can define, a potential associated with magnetostatic field B. In fact, the magnetic potential could be a Scalar V_m (or) Vector A.

→ To define V_m and A involves two important identities.

$$\boxed{\begin{array}{ll} \text{i.} & \nabla \times (\nabla V) = 0 \\ \text{ii.} & \nabla \cdot (\nabla \times A) = 0 \end{array}}$$

<u>Def</u>: The scalar magnetic potential is defined as the line integral of magnetic field intensity along the given path, where the current density is zero

Mathematically $\boxed{V_m = -\int H \cdot dl}$

→ The scalar magnetic potential b/w two points A & B in the magnetic field is $V_{m A,B} = -\int_B^A H \cdot dl$ Amp.

→ The negative gradient of the scalar magnetic ~~field~~ potential is equal to the magnetic field intensity at which the current density $J=0$.

$\therefore H = -\nabla V_m$ if $J=0$.

taking curl on both sides

$\nabla \times H = -\nabla \times \nabla V_m$

→ But we know that, the curl of gradient of any scalar is always zero.

$\therefore \nabla \times H = -\nabla \times \nabla V_m = 0. \longrightarrow (1).$

→ from Amperes circuit law (or) maxwell's 3rd equation

$\nabla \times H = J \longrightarrow (2)$

equating eq(1) & (2)

$$\boxed{J = 0.}$$

→ Therefore the scalar magnetic potential V_m exists in the region where the current density is zero.

Laplace equation for scalar Potential :

We know that $\nabla \cdot B = 0$

$\nabla \cdot \mu H = 0$

Since $\mu \neq 0$, then $\nabla \cdot H = 0$

But $H = -\nabla V_m$, then $\nabla \cdot (-\nabla V_m) = 0$

$\nabla^2 V_m = 0$

$$\boxed{\therefore \nabla^2 V_m = 0} \quad \text{for } J = 0.$$

Application :

→ Scalar magnetic potential is used to find the magnetic field intensity and flux in the region, where there is no current exist.

→ It is also used to plot the flux lines for permanent magnets.

② Magnetic Vector Potential : (A)

In a given magnetic field, the curl of the vector magnetic-Potential A is equal to the magnetic flux density. i.e.,

$$\boxed{B = \nabla \times A}$$

where A is vector magnetic potential, its units are Wb/m

Vector Magnetic Potential for Line current :

Vector magnetic potential at a Point 'P' in the magnetic field is proportional to the differential current element $I\,dl$ flowing through the conductor and inversely proportional to the distance vector from the current element to the field Point P. It is denoted as A.

It can be expressed as $$\boxed{A = \int_L \frac{\mu I\,dl}{4\pi R}}$$

Proof:

→ from Biot-Savarts law we know that for a differential line current element Idl,

$$H = \int_L \frac{Idl \times a_R}{4\pi R^2}$$

→ Relation b/w magnetic flux density (B) and magnetic field intensity (H) is

$$B = \mu_0 H = \frac{\mu}{4\pi}\int_L \frac{Idl \times a_R}{R^2}$$

$$B = \frac{\mu}{4\pi}\int_L Idl \times \left(\frac{a_R}{R^2}\right) = \frac{-\mu}{4\pi}\int_L Idl \times \nabla\left(\frac{1}{R}\right) \longrightarrow (1)$$

$\nabla\left(\frac{1}{R}\right) = \frac{-1}{R^2} a_R$

$-\nabla\left(\frac{1}{R}\right) = \frac{a_R}{R^2}$

→ From vector Identity $\nabla \times VA = \nabla V \times A + V \nabla \times A$

Here $V = \frac{1}{R}$ and $A = Idl$

$$\Rightarrow \nabla V \times A = \nabla \times VA - V\nabla \times A$$

$$\Rightarrow \nabla\left(\frac{1}{R}\right) \times Idl = \nabla \times \left(\frac{Idl}{R}\right) - \frac{1}{R}\nabla \times Idl$$

$$\Rightarrow \nabla\left(\frac{1}{R}\right) \times Idl = \nabla \times \left(\frac{Idl}{R}\right)$$

$\because$ Since $\nabla \times Idl = 0$

$$\Rightarrow Idl \times \nabla\left(\frac{1}{R}\right) = -\nabla \times \left(\frac{Idl}{R}\right) \longrightarrow (2)$$

→ Substitute eq (2) in (1)

$$B = \frac{-\mu}{4\pi}\int_L -\nabla \times \left(\frac{Idl}{R}\right) = \nabla \times \int_L \frac{\mu Idl}{4\pi R} \longrightarrow (3)$$

Vector magnetic Potential $B = \nabla \times A \longrightarrow (4)$

Compare (3) & (4) then $\boxed{A = \int_L \frac{\mu Idl}{4\pi R}}$

(OR)

$$B = \nabla \times A \Rightarrow \mu H = \nabla \times A$$

$$\mu \int_L \frac{Idl \times a_R}{4\pi R^2} = \nabla \times A \Rightarrow \mu \int_L \frac{Idl}{4\pi R} \times \frac{a_R}{R} = \nabla \times A$$

(or) $$\int_L \frac{\mu Idl}{4\pi R} \times \frac{a_R}{R} = +\nabla \times A \Rightarrow -\frac{a_R}{R} \times \int_L \frac{\mu Idl}{4\pi R} = \nabla \times A$$

$B = \mu H$

$H = \int_L \frac{Idl \times a_R}{4\pi R^2}$

$A \times B = -B \times A$

$\nabla\left(\frac{1}{R}\right) = \frac{-1}{R^2} a_R$

$\nabla = -\frac{a_R}{R}$

$-\nabla = \frac{a_R}{R}$

Comparing LHS & RHS $\boxed{A = \int_L \frac{\mu Idl}{4\pi R}}$

→ Vector magnetic potential for a line current is

$$A = \int_L \frac{\mu I dl}{4\pi R} \quad Wb/m$$

→ Similarly vector magnetic potential for a surface current is

$$A = \int_S \frac{\mu K ds}{4\pi R} \quad Wb/m$$

→ Similarly vector magnetic potential for a volume current is

$$A = \int_v \frac{\mu J dv}{4\pi R} \quad Wb/m.$$

Poisson's Equation for Vector Magnetic Potential :

→ We know that, from Ampere's circuit law

$$\nabla \times H = J \Rightarrow \nabla \times \frac{B}{\mu} = J \Rightarrow \nabla \times B = \mu J$$

→ But $B = \nabla \times A$ then

$$\nabla \times \nabla \times A = \mu J$$

→ From the vector Identity $\nabla^2 A = \nabla(\nabla \cdot A) - \nabla \times \nabla \times A$

* We know that A is a vector field due to time invarient current. For time invarient fields if curl exists and divergence vanishes. Therefore

$\nabla \cdot A = 0$ Then $\nabla^2 A = 0 - \nabla \times \nabla \times A \Rightarrow \nabla \times \nabla \times A = -\nabla^2 A = \mu J$

$$\boxed{\therefore \nabla^2 A = -\mu J}$$

→ This is the Poissons equation for magnetic field.

If $J = 0$ then $\nabla^2 A = 0$. This becomes a laplace's equation.

Applications :

1. Vector magnetic potential is used to obtain the radiation characteristics of antennas
2. It is used to find near field and far fields of antennas.

Problem (7.7) Given the magnetic vector Potential $A = -\frac{\rho^2}{4} a_z$ wb/m.

(a) Calculate the magnetic flux density

(b) Total magnetic flux crossing the Surface $\phi = \frac{\pi}{2}$, $1m \leq \rho \leq 2m$, $0 \leq z \leq 5m$.

Sol Given $A = \frac{-\rho^2}{4} a_z$ wb/m ; use cylindrical curl operation.

(a) we have $B = \nabla \times A$

$$\therefore \nabla \times A = \frac{1}{\rho} \begin{vmatrix} a_\rho & \rho a_\phi & a_z \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ 0 & 0 & -\frac{\rho^2}{4} \end{vmatrix}$$

$$\nabla \times A = \frac{1}{\rho}\left[a_\rho\left[\frac{\partial}{\partial \phi}\left(-\frac{\rho^2}{4}\right) - 0\right] - \rho a_\phi\left[\frac{d}{\partial \rho}\left(\frac{-\rho^2}{4}\right) - 0\right] + a_z[0-0] \right]$$

$$= \frac{1}{\rho}\left[0 + \rho a_\phi \cdot \frac{1}{4} 2\rho + 0\right]$$

$$= \frac{1}{\rho}\left[\frac{\rho^2}{2} a_\phi\right] = \frac{\rho}{2} a_\phi$$

$$\boxed{\therefore B = \nabla \times A = \frac{\rho}{2} a_\phi} \quad \text{wb/m}^2$$

(b) $dS = \partial\rho \, dz \, a_\phi$ differential Surface area along a_ϕ direction

$$\psi = \oint_S B \cdot dS = \int_S \frac{\rho}{2} a_\phi \cdot \partial\rho \, dz \, a_\phi = \int_S \frac{\rho}{2} \partial\rho \, \partial z$$

$$\psi = \frac{1}{2} \int_1^2 \rho \, d\rho \int_0^5 dz = \frac{1}{2}\left[\frac{\rho^2}{2}\right]_1^2 [z]_0^5 = \frac{1}{4}[4-1][5]$$

$$\psi = \frac{1}{4}(3)(5) = \frac{15}{4} = 3.75 \text{ wb} \qquad \boxed{\therefore \psi = 3.75 \text{ Wb}}$$

Problem A current distribution gives rise to the vector magnetic potential $A = x^2 y a_x + y^2 x a_y - 4xyz a_z$ Wb/m. Calculate (a) B at $(-1, 2, 5)$

(b) the flux through the Surface defined by $z = 1$, $0 \leq x \leq 1$, $-1 \leq y \leq 4$.

Ans: (a) $20 a_x + 40 a_y + 3 a_z$ wb/m²

(b) 20 wb.

YVNR

MF-12

Force due to Magnetic Field :

→ There are three ways in which force due to magnetic field can be experienced.

1. Force on a charged Particle.
2. Force on a current element.
3. Force between two current elements.

1. Force on a charged Particle :

→ If there is a charge (or) a moving charge Q, in an electric field E, there exists a force on the charge. This force is given by

$$F_e = QE$$

→ If a charge Q, moving with a velocity u is placed in a magnetic field H, then there exist a force on the charge. This force is given by $F_m = Q(u \times B)$

→ If the charge Q is placed in both electric and magnetic fields, then the force on the charge is

$$F = QE + Q(u \times B)$$

$$\boxed{F = Q[E + u \times B]}$$

This equation is known as Lorentz Force equation.

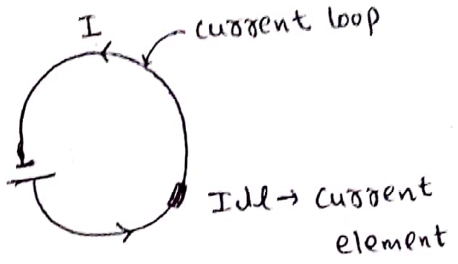
→ Lorentz force equation relates mechanical force to electrical force.

→ If the mass of the charged particle moving in E & B fields is m, by Newton's Second law of motion

$$\boxed{F = ma = m\frac{du}{dt} = Q[E + u \times B]}$$

→ This equation is used to determine the motion of charged particle in E and B fields.

2. Force on a current element :



→ Force on a current element Idl with in the magnetic field B can be obtained from the formula

$$\boxed{F = \oint_L I dl \times B}$$

Proof :

→ The current I is defined as the rate of flow of charge Q.

i.e., $I = \frac{dQ}{dt}$

$$\Rightarrow I\,dl = \frac{dQ}{dt} \cdot dl$$

$$\Rightarrow I dl = dQ \cdot \frac{dl}{dt}$$

→ But we know that rate of change of displacement is called velocity

i.e $\frac{dl}{dt} = u$

$$\therefore I dl = dQ \cdot u$$

$$I dl \times B = dQ\,(u \times B)$$

$$I dl \times B = dF$$

$$\therefore dF = I dl \times B \Rightarrow \boxed{F = \oint_L I dl \times B}$$

→ Force on a line current element is $F = \int_L I dl \times B$

→ Force on a Surface current element is $F = \int_S K ds \times B$

→ Force on a volume current element is $F = \int_V J dv \times B$

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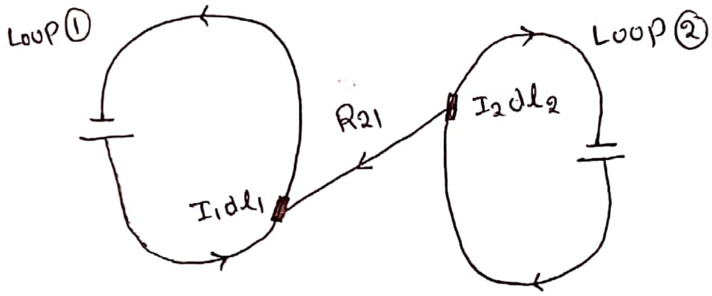
3. Force between two current elements :

Ampere's force law States that the force between two current elements $I_1 dl_1$ and $I_2 dl_2$ placed at a distance R_{21} is given by

$$\boxed{F_{21} = \frac{\mu I_1 I_2}{4\pi} \oint_{L_1} \oint_{L_2} \frac{dl_1 \times dl_2 \times a_{R_{21}}}{R_{21}^2}}$$

Proof :



→ Let us consider the force between two current elements $I_1 dl_1$ and $I_2 dl_2$.

→ According to the Biot-Savart's law, both current elements produces magnetic fields.

→ But we know the force on a current element Idl with in the magnetic field B, is $F = \int_L Idl \times B$

$$\Rightarrow dF = Idl \times B$$

→ Force on current element $I_1 dl_1$ due to the magnetic field B_2, produced by the current element $I_2 dl_2$ is

$$dF_{21} = I_1 dl_1 \times B_2$$

$$= I_1 dl_1 \times \mu H_2$$

$$= I_1 dl_1 \times \mu \int_{L_2} \frac{I_2 dl_2 \times a_{R_{21}}}{4\pi R_{21}^2}$$

$$dF_{21} = \frac{\mu I_1 I_2}{4\pi} \int_{L_2} \frac{dl_1 \times dl_2 \times a_{R_{21}}}{R_{21}^2}$$

$$\therefore F_{21} = \int_{L_1} \frac{\mu I_1 I_2}{4\pi} \int_{L_2} \frac{dl_1 \times dl_2 \times a_{R_{21}}}{R_{21}^2}$$

$$\boxed{F_{21} = \frac{\mu I_1 I_2}{4\pi} \int_{L_1} \int_{L_2} \frac{dl_1 \times dl_2 \times a_{R_{21}}}{R_{21}^2}}$$

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Inductance :

(1) The inductance of the inductor is defined as the ratio of total flux linkage to the current flowing through the circuit.

$$\boxed{L = \frac{\lambda}{I} = \frac{N\psi}{I}}$$ Henry.

→ The inductance is also known as self inductance.

Flux Linkage :

It is defined as the product of number of turns N and the total flux ψ linking each of the turn.

$$\boxed{\lambda = N.\psi}$$ web. turn

Inductance of Solenoid :

Solenoid is a conductor (or) rod of length l made up of magnetic material with coil of N-turns wounded on the rod (or) conductor.

$$\boxed{L = \frac{N^2 \mu S}{l}}$$ Henry

Inductance of Toroid :

Toroid is a solid ring of radius R made up of magnetic material with coil of N-turns wounded on the ring

$$\boxed{L = \frac{N^2 \mu S}{2\pi R}}$$ Henry.

S → Cross sectional area of Solenoid (or) Toroid.

μ → Permeability.

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MF-17

Magnetic Energy :

→ The energy stored in magnetic field by an inductor is given by

$$\boxed{W_m = \frac{1}{2} L I^2}$$

where $L \rightarrow$ inductance
$I \rightarrow$ applied current

Proof:

→ If a current I is applied on a coil having N turns, it produces a magnetic field with flux ψ.

The voltage induced across the coil is

$$V = L \frac{dI}{dt}$$

→ We know that power P is $P = VI = L\frac{dI}{dt} \cdot I = LI \frac{dI}{dt} \longrightarrow (1)$

→ But we know that power P is $P = \frac{W_m}{t} \Rightarrow W_m = P.t = \int P.dt \longrightarrow (2)$

Substitute eq (1) in eq (2)

$$W_m = \int LI \frac{dI}{dt} . dt = \int LI . dI = L \int I dI = \frac{LI^2}{2}$$

$$\boxed{\therefore W_m = \frac{1}{2} L I^2}$$ Jouls.

Energy density stored in magnetic field :

→ Eenergy density is defined as the energy per unit volume.

→ The energy density stored in magnetic field of an inductor is given by $\boxed{W = \frac{1}{2} BH}$

Proof :

→ The energy stored in magnetic field by an inductor is given by

$$W_m = \frac{1}{2} L I^2$$

→ Consider a solenoid, its inductance is

$$L = \frac{\mu N^2 S}{l}$$

Then $W_m = \frac{1}{2}\frac{\mu N^2 S}{l} I^2 \times \left(\frac{l}{l}\right)$

$W_m = \frac{1}{2}\mu \left(\frac{NI}{l}\right)^2 lS$

$\boxed{W_m = \frac{1}{2}\mu H^2 v}$ Joules

Volume $v = lS$ m^3

$l \cdot H = NI$

$H = \frac{NI}{l}$

→ The energy density

$$W = \frac{W_m}{v} = \frac{\frac{1}{2}\mu H^2 v}{v} = \frac{1}{2}\mu H^2$$

$$\therefore W = \frac{1}{2}\mu H \cdot H = \frac{1}{2} B \cdot H$$

$\boxed{\therefore W = \frac{1}{2} BH}$ (or) $\boxed{W = \frac{1}{2}\frac{B^2}{\mu}}$ Jouls/m^3

→ If dv is the differential volume then the energy stored is

$\boxed{W_m = \int W\, dv = \frac{1}{2}\int_v B \cdot H\, dv}$ Joules.